

Curvature Inspired Cosmological Scenario

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Abstract Using modified gravity with non-linear terms of curvature, R^2 and $R^{(2+r)}$ (with r being a positive real number and R being the scalar curvature), cosmological scenario, beginning at the Planck scale, is obtained. Here a unified picture of cosmology is obtained from $f(R)$ -gravity. In this scenario, universe begins with power-law inflation followed by deceleration and acceleration in the late universe as well as possible collapse of the universe in future. It is different from $f(R)$ -dark energy models with non-linear curvature terms assumed as dark energy. Here, dark energy terms are induced by linear as well as non-linear terms of curvature in Friedmann equation being derived from modified gravity. It is also interesting to see that, in this model, dark radiation and dark matter terms emerge spontaneously from the gravitational sector. It is found that dark energy, obtained here, behaves as quintessence in the early universe and phantom in the late universe. Moreover, analogous to brane-tension in brane-gravity inspired Friedmann equation, a tension term λ arises here being called as cosmic tension. It is found that, in the late universe, Friedmann equation (obtained here) contains a term $-\rho^2/2\lambda$ (ρ being the phantom energy density) analogous to a similar term in Friedmann equation with loop quantum effects, if $\lambda > 0$ and brane-gravity correction when $\lambda < 0$.

Keywords Modified gravity · Dark energy · Acceleration and deceleration of the universe

1 Introduction

Cosmology was revolutionized by observations made during last few years [1–3]. These observations show conclusive evidence for acceleration in the late universe, which is still a challenge for cosmologists. Theoretically, it is found that dark energy (DE) violating *strong*

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energy condition (SEC) or *weak energy condition* (WEC) is responsible for it. So, in the recent past, many DE models were proposed to explain the *late cosmic acceleration*. A comprehensive review of these models is available in [4]. Later on, it was realized that even non-linear terms of curvature R^{-n} with $n > 0$ also could be used as DE [5, 6]. Although this model explained late cosmic acceleration, it exhibited instability and failed to satisfy solar system constraints. It was improved further by Nojiri and Odintsov taking different forms of $f(R)$ for DE. These improved models satisfied solar system constraints exhibiting late cosmic acceleration for small curvature and early inflation for large curvature. Thus, in $f(R)$ -dark energy models, non-linear curvature terms are considered as an alternative for DE [7, for detailed review]. Recently, in [8, 9], it is shown that $f(R)$ -dark energy models with dominating powers of R for large or small R can not yield viable cosmology as results contradict the standard model and do not satisfy Wilkinson Microwave Anisotropy Probe (WMAP) results, though these models pass solar system constraints and explain late acceleration. Moreover, it is also shown that the most popular model with $f(R) = R + \alpha R^m + \beta R^{-n}$ ($m > 0, n > 0$) considered in [7] is unable to produce matter in the late universe prior to the beginning of late acceleration [8, 9].

In what follows, it is aimed to get a viable cosmology consistent with WMAP from $f(R)$ -gravity, where $f(R) = R/16\pi G +$ powers of R , not from $f(R)$ -dark energy models [10–13] discussed above. There is a crucial difference between the two. In the latter case, which is criticised in [8, 9], non-linear terms of curvature are treated as dark energy terms. On the contrary, in the former case, neither linear nor non-linear term is considered as dark energy. In the present model, it is important to see that DE terms are induced by linear (Einstein-Hilbert term) as well as non-linear terms R^2 and $R^{(2+r)}$ in the action. In $f(R)$ -dark energy models, dark energy terms depend on $f(R)$ terms and its derivative $F = df/dR$. In the former case, induced DE terms depend on the scale factor $a(t)$ of the homogeneous and flat of Friedmann-Robertson-Walker (FRW) model of the universe.

In this paper, the $f(R)$ -gravity based modified Friedmann equations are derived in the early and late universe taking small and large $a(t)$ respectively. Contrary to $f(R)$ -dark energy models dominance of non-linear terms of curvature is not taken here for small and large R as, in the present model, DE terms emerge as imprints of both linear term as well as powers of R . In $f(R)$ -dark energy models, action contains lagrangians for matter and radiation [7–9]. It is interesting to see that, in the present $f(R)$ -gravity model of cosmology, dark radiation and dark matter terms emerge spontaneously [12, 13].

Friedmann equation (FE), obtained here, gives cosmic dynamics, where some terms emerge having forms different from known forms of energy density (radiation or matter) and violate SEC in the case of early universe and WEC in the case of late universe. So, these terms are recognized as curvature induced DE. Thus, this approach of getting DE from curvature is *different* from the approach of [7–9]. It is also interesting to see that, in the case of late universe, FE contains square of DE density with negative sign analogous to a similar term in Friedmann equation with loop quantum effects [14].

In what follows, it is interesting to see that radiation and matter terms in FE (obtained here) emerge spontaneously if $r = 3$ and $n = 1/4$. In [7], radiation and matter terms do not emerge from the gravitational sector. It is important to mention that, here, theory itself ignores the cases of $r = 1, 2$ i.e. theory suggests that non-linear of R should be R^2 and R^5 . In [7], a certain form of scale factor $a(t)$ is assumed and, later on, conditions are obtained for assumed $a(t)$ consistent with experiments and satisfying stability criterion. Here, $a(t)$ is *not* assumed, but it is derived solving Friedmann equations in the different stages of the universe. This is another important *difference* in approach of this paper compared to [7]. All these results are obtained from modified gravity without using any exotic matter or field. This approach is adapted in [10–13] also.

In [13], a unified picture of the universe, from early inflationary stage to late acceleration and deceleration driven by dark radiation and dark matter between these two stages, is obtained taking linear and non-linear terms of curvature as well as a different scalar. But, in this paper, the same result is obtained from curvature terms only.

Investigations, given below, show that, originating at the Planck scale, the universe inflates for a very short period followed by deceleration driven by curvature-induced dark radiation and subsequently by dark matter. In the very late universe, around 12.86 Gyr, a transition from *deceleration* to *acceleration* takes place. Further, it is found that late acceleration will continue upto 58.48 Gyr. This is an epoch for transition from *acceleration* to *deceleration*. At this epoch, acceleration will stop and deceleration, driven by matter, will resume [13, 16]. The decelerated expansion will continue upto the time $\sim 1.05 \times 10^{156}$ Gyr. By this time, the universe will have maximum expansion. So, it is natural to think that the universe will retrace back and contract. Results show that, due to contraction, universe will collapse by the time $\sim 1.3 \times 10^{156}$ Gyr.

In [17], quintessence DE in the early universe and phantom DE in the late universe have been considered taking non-gravitational scalar field (curvature independent scalar field) as DE source. The present paper is different from [17] in the sense that here we have gravitational origin of quintessence and phantom DE as it is obtained in references [10–13].

Natural units ($k_B = \hbar = c = 1$) (where k_B, \hbar, c have their usual meaning. GeV is used as a fundamental unit and we have $1 \text{ GeV}^{-1} = 6.58 \times 10^{-25} \text{ sec.}$ and $1 \text{ GeV} = 1.16 \times 10^{13} \text{ K.}$

2 Action for $f(R)$ -gravity and Friedmann Equations

Here action is taken as

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \alpha R^2 + \beta R^{(2+r)} \right], \tag{2.1}$$

where $G = M_p^{-2}$ ($M_p = 10^{19}$ GeV is the Planck mass), α is a dimensionless coupling constant, β is a constant having dimension $(\text{mass})^{(-2r)}$ (as R has mass dimension 2) with r being a positive real number.

Using the condition $\delta S / \delta g^{\mu\nu} = 0$, action (2.1) yields field equations

$$\begin{aligned} & \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \alpha \left(2 \nabla_\mu \nabla_\nu R - 2 g_{\mu\nu} \square R - \frac{1}{2} g_{\mu\nu} R^2 + 2 R R_{\mu\nu} \right) \\ & + \beta(2+r) (\nabla_\mu \nabla_\nu R^{(1+r)} - g_{\mu\nu} \square R^{(1+r)}) + \frac{1}{2} \beta g_{\mu\nu} R^{(2+r)} \\ & - \beta(2+r) R^{(1+r)} R_{\mu\nu} = 0, \end{aligned} \tag{2.2}$$

where ∇_μ stands for the covariant derivative.

Taking trace of (2.2), it is obtained that

$$-\frac{R}{16\pi G} - 6\alpha \square R - 3\beta(2+r) \square R^{(1+r)} + \beta r R^{(2+r)} = 0 \tag{2.3}$$

with

$$\square = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right). \tag{2.4}$$

In (2.3)

$$\square R^{(1+r)} = (1+r)[R^r \square R + r R^{(r-1)} \nabla^\mu R \nabla_\mu R], \tag{2.5}$$

From (2.3) and (2.5)

$$\begin{aligned} &-\frac{R}{16\pi G} - [6\alpha + 3\beta(1+r)(2+r)R^r] \square R - 3\beta r(1+r)(2+r)R^{(r-1)} \nabla^\mu R \nabla_\mu R \\ &+ \beta r R^{(2+r)} = 0. \end{aligned} \tag{2.6}$$

In (2.6), $[6\alpha + 3\beta(1+r)(2+r)R^r]$ emerges as a coefficient of $\square R$ due to presence of terms αR^2 and $\beta R^{(2+r)}$ in the action (2.1). If $\alpha = 0$, effect of R^2 vanishes and effect of $R^{(2+r)}$ is switched off for $\beta = 0$. So, like [13] an *effective* scalar curvature \tilde{R} is defined as

$$\gamma \tilde{R}^r = [6\alpha + 3\beta(1+r)(2+r)R^r], \tag{2.7}$$

where γ is a constant having dimension $(\text{mass})^{-2r}$ being used for dimensional correction.

Using (2.7) in (2.6), we have

$$\begin{aligned} &\frac{1}{16\pi G} Y^{1/r} - (\gamma/r) \tilde{R}^r Y^{(1/r-2)} [Y \square Y + (1/r - 1) \nabla^\mu Y \nabla_\mu Y] \\ &- 3(\beta/r)(1+r)(2+r) Y^{(1/r-1)} \nabla^\mu Y \nabla_\mu Y + \beta r Y^{(2+r)/r} = 0, \end{aligned} \tag{2.8}$$

where

$$Y = R^r = \frac{\gamma \tilde{R}^r - 6\alpha}{3\beta(1+r)(2+r)}. \tag{2.9a}$$

Equation (2.8) is simplified as

$$\begin{aligned} &\frac{1}{16\pi G} Y - (\gamma/r) \tilde{R}^r [\square Y + (1/r - 1) Y^{-1} \nabla^\mu Y \nabla_\mu Y] \\ &- 3(\beta/r)(1+r)(2+r) \nabla^\mu Y \nabla_\mu Y + \beta r Y^{(1/r+2)} = 0. \end{aligned} \tag{2.9b}$$

Using (2.9a) in (2.9b), it is obtained that

$$\begin{aligned} &-\frac{r}{16\pi G \gamma} \left[\frac{6\alpha}{\gamma \tilde{R}^r} - 1 \right] + \square \tilde{R}^r - (1/r - 1) \frac{\gamma}{[6\alpha - \gamma \tilde{R}^r]} \nabla^\mu \tilde{R}^r \nabla_\mu \tilde{R}^r + \tilde{R}^{-r} \nabla^\mu \tilde{R}^r \nabla_\mu \tilde{R}^r \\ &+ [3\beta^2 r(1+r)(2+r)/\gamma^2] \tilde{R}^r \left[\frac{\gamma \tilde{R}^r - 6\alpha}{3\beta(1+r)(2+r)} \right]^{(1/r+2)} = 0. \end{aligned} \tag{2.10}$$

Equation (2.10) is re-written as

$$\begin{aligned} &-\frac{1}{16\pi G} \frac{1}{\gamma \tilde{R}^{r-1}} \left[\frac{6\alpha}{\gamma \tilde{R}^r} - 1 \right] + \square \tilde{R} + (r - 1) \tilde{R}^{-1} \nabla^\mu \tilde{R} \nabla_\mu \tilde{R} \\ &- (1 - r) \frac{\gamma \tilde{R}^{r-1}}{6\alpha - \gamma \tilde{R}^r} \nabla^\mu \tilde{R} \nabla_\mu \tilde{R} + r \tilde{R}^{-1} \nabla^\mu \tilde{R} \nabla_\mu \tilde{R} \\ &+ r(2+r)/\gamma^2] \tilde{R}^{2r-1} \left[\frac{\gamma \tilde{R}^r - 6\alpha}{3\beta(1+r)(2+r)} \right]^{(1/r+2)} = 0. \end{aligned} \tag{2.11}$$

Experimental evidences [18–22] support spatially homogeneous flat model of the universe

$$dS^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2] \tag{2.12}$$

with $a(t)$ being the scale factor.

For $a(t)$, being the power-law function of cosmic time, $\tilde{R} \sim a^{-n}$. For example, $\tilde{R} \sim a^{-3}$ for matter-dominated model. So, there is no harm in taking

$$\tilde{R} = \frac{A}{a^n}, \tag{2.13}$$

where $n > 0$ is a real number and A is a constant with mass dimension 2.

Connecting (2.11) and (2.13), it is obtained that

$$\begin{aligned} \frac{\ddot{a}}{a} + \left[2 - n - n(r - 1) + \frac{n(1 - r)\gamma A^r a^{-nr}}{6\alpha - \gamma A^r a^{-nr}} - nr \right] \left(\frac{\dot{a}}{a} \right)^2 &= \frac{a^{nr}}{16\pi G \gamma A^r} \left[\frac{6\alpha a^{nr}}{\gamma A^r} - 1 \right] \\ &- \frac{\beta^{-1/3}}{n(\gamma A^r a^{-nr})^2 [3r(1 + r)(2 + r)]^{1+1/r}} [6\alpha - \gamma A^r a^{-nr}]^{2+1/r}, \end{aligned} \tag{2.14}$$

taking $(-\beta)^{-1/3} = -\beta^{-1/3}$ and ignoring complex roots as these roots lead to unphysical situations. Now, we have following two cases.

Case 1: The Early Universe

In this case, $a(t)$ is very small, so (2.14) is approximated as

$$\begin{aligned} \frac{\ddot{a}}{a} + [2 - n - nr] \left(\frac{\dot{a}}{a} \right)^2 &\simeq - \frac{\beta^{-1/3}}{n(\gamma A^r a^{-nr})^2 [3r(1 + r)(2 + r)]^{1+1/r}} \\ &\times [6\alpha - \gamma A^r a^{-nr}]^{2+1/r} \end{aligned} \tag{2.15}$$

as

$$\frac{\gamma A^r a^{-nr}}{6\alpha - \gamma A^r a^{-nr}} \approx -1. \tag{2.16}$$

Integration of (2.15) leads to

$$\begin{aligned} \left(\frac{\dot{a}}{a} \right)^2 &= \frac{B}{a^{(2+2M)}} - \frac{2\beta^{-1/3}}{n(\gamma A^r)^2 [3r(1 + r)(2 + r)]^{1+1/r} a^{(2+2M)}} \\ &\times \int a^{(1+2M+2nr)} [6\alpha - \gamma A^r a^{-nr}]^{2+1/r} \end{aligned} \tag{2.17}$$

with

$$M = 2 - n - nr. \tag{2.18}$$

Case 2: The Late Universe

In this case, $a(t)$ is large, so (2.14) is approximated as

$$\begin{aligned} \frac{\ddot{a}}{a} + \left[2 - n - n(r - 1) - nr \right] \left(\frac{\dot{a}}{a} \right)^2 &\simeq \frac{a^{nr}}{16\pi G \gamma A^r} \left[\frac{6\alpha a^{nr}}{\gamma A^r} - 1 \right] \\ &- \frac{\beta^{-1/3}}{n(\gamma A^r a^{-nr})^2 [3r(1 + r)(2 + r)]^{1+1/r}} (6\alpha)^{2+1/r} [a^{2nr} - (2 + 1/r)\gamma A^r a^{nr}] \end{aligned} \tag{2.19a}$$

as

$$\frac{\gamma A^r a^{-nr}}{6\alpha - \gamma A^r a^{-nr}} \approx 0$$

for large scale factor a . So, (2.19a) is re-written as

$$\frac{\ddot{a}}{a} + [2 - 2nr] \left(\frac{\dot{a}}{a}\right)^2 = Da^{nr} - Ea^{2nr}, \tag{2.19b}$$

where

$$D = \left(\frac{6\alpha}{\gamma A^r}\right) \left[\frac{1}{16\pi Gn} - (2 + 1/r) \frac{[3r(1+r)(2+r)]^{-1-1/r}}{n} \left(\frac{6\alpha}{\gamma A^r}\right) \right] \tag{2.20a}$$

and

$$E = \left(\frac{6\alpha}{\gamma A^r}\right)^2 \left[\frac{1}{16\pi Gn} - \frac{[3r(1+r)(2+r)]^{-1-1/r}}{n} \left(\frac{6\alpha}{\gamma A^r}\right) \right]. \tag{2.20b}$$

Equation (2.19b) is integrated to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{B}{a^{(2+2N)}} + \frac{2D}{(2 + 2N + nr)} a^{nr} \left[1 - \frac{E(2 + 2N + nr)}{D(2 + 2N + 2nr)} a^{nr} \right] \tag{2.21}$$

with

$$N = 2 - 2nr. \tag{2.22}$$

Further, it is found that if $M = 1$, the first term on r.h.s. (right hand side) of (2.17) gives radiation. Moreover, if $N = 1/2$ the first term of r.h.s. of (2.21) gives matter. So, using $M = 1$ in (2.18) and $N = 1/2$ in (2.22), it is obtained that

$$nr = \frac{3}{4}, \tag{2.23}$$

$$n = \frac{1}{4} \tag{2.24}$$

and

$$r = 3. \tag{2.25}$$

3 Power-Law Inflation Followed by Deceleration in the Early Universe

The approximated Friedmann equation (2.17), in the case of the early universe, looks like

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{B}{a^4} - \frac{8\beta^{-1/3}}{(\gamma A^3)^2 [180]^{4/3} a^4} \int a^{9/2} [6\alpha - \gamma A^3 a^{-3/4}]^{7/3} \tag{3.1}$$

using definitions of M and N as well as (2.24) and (2.25). In (3.1),

$$\begin{aligned} \int a^{9/2} [6\alpha - \gamma A^3 a^{-3/4}]^{7/3} &= \left[\frac{2}{11} a^{11/2} \{6\alpha - \gamma A^3 a^{-3/4}\}^{7/3} \right] - \frac{7}{22} \gamma A^3 \\ &\times \int a^{15/4} \{6\alpha - \gamma A^3 a^{-3/4}\}^{4/3} da. \end{aligned} \tag{3.2a}$$

It is noted that for

$$a < (\gamma A^3/6\alpha)^{4/3} = a_c, \tag{3.2b}$$

terms within bracket and the integral on the right hand side of (3.2a) are of the order of $a^{15/4}$.

So,

$$\int a^{9/2}[6\alpha - \gamma A^3 a^{-3/4}]^{7/3} \approx \left[\frac{2}{11} a^{11/2} \{6\alpha - \gamma A^3 a^{-3/4}\} \right]^{7/3}. \tag{3.2c}$$

Thus, using (3.2), (3.1) is approximated as

$$\left(\frac{\dot{a}}{a}\right)^2 \approx \frac{B}{a^4} - \frac{16\beta^{-1/3}}{11(\gamma A^3)^{-1/3}[180]^{4/3}} a^{3/2} [a_c^{-3/4} - a^{-3/4}]^{7/3}. \tag{3.3}$$

It is interesting to see that a radiation density term B/a^4 emerges spontaneously. This type of a term, being called dark radiation, emerges in brane-gravity inspired Friedmann equation too. So, analogous to brane-gravity, here also B/a^4 is called dark radiation. Other terms on r.h.s. of (3.3) are caused by linear as well as non-linear terms of curvature in the action (2.1). These terms also constitute energy density term

$$\rho_{de}^{qu} = \frac{3}{8\pi G} \left[\frac{16\beta^{-1/3}}{11(\gamma A^3)^{-1/3}[180]^{4/3}} a^{3/2} \right] \left[a^{-3/4} - \frac{6\alpha}{\gamma A^3} \right]^{7/3} \tag{3.4}$$

(taking real root of $(-1)^{-1/3}$ as above) satisfying the conservation equation

$$\dot{\rho}_{de} + 3\frac{\dot{a}}{a}(\rho_{de} + p_{de}) = 0. \tag{3.5}$$

Connecting (3.4) and (3.5), equation of state (EOP) is obtained as

$$p_{de}^{qu} = -\frac{3}{2}\rho_{de}^{qu} + \frac{7}{12}f[a^{-3/4} - a_c^{-3/4}]^{4/3}, \tag{3.6a}$$

where

$$f = \frac{3}{8\pi G} \frac{16\beta^{-1/3}}{11(\gamma A^3)^{-1/3}[180]^{4/3}}, \tag{3.6b}$$

(3.6a) is the scale factor-dependent equation of state parameter, valid for $a_p \leq a(t) < a_c$. Such an equation of state parameter is obtained in [12] also. It yields

$$\rho_{de}^{qu} + p_{de}^{qu} > 0 \quad \text{and} \quad \rho_{de}^{qu} + 3p_{de}^{qu} < 0,$$

for $a_p \leq a(t) \leq a_c$. It shows that DE, having energy density (3.4) mimics *quintessence dark energy* [10–13].

Here investigations start at the Planck scale, where DE density is obtained around 10^{75} GeV⁴. So, (3.4) is obtained as

$$\rho_{de}^{qu} = F a^{3/2} [a^{-3/4} - a_c^{-3/4}]^{7/3} \tag{3.7a}$$

with

$$F = 10^{75} a_p^{-3/2} [a_p^{-3/4} - a_c^{-3/4}]^{-7/3}. \tag{3.7b}$$

Thus, (3.6b) and (3.7b) imply

$$f = F. \tag{3.7c}$$

Connecting (3.3) and (3.7a), it is obtained that

$$\left(\frac{\dot{a}}{a}\right)^2 \simeq \frac{B}{a^4} + \frac{8\pi \times 10^{37}}{3} \left(\frac{a}{a_P}\right)^{3/2} \left[\frac{a^{-3/4} - a_c^{-3/4}}{a_P^{-3/4} - a_c^{-3/4}}\right]^{7/3} \tag{3.8}$$

using $G = M_P^{-2} = 10^{-38} \text{ GeV}^{-2}$.

It means that the universe is driven by radiation for $a \geq a_c$. Moreover, (3.7a) shows that $\rho_{\text{de}}^{\text{qu}}$ vanishes at $a = a_c$ and for $a_P < a(t) < a_c$, cosmic dynamics is given by

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &\simeq \frac{8\pi \times 10^{37}}{3} \left(\frac{a}{a_P}\right)^{3/2} \left[\frac{a_c^{-3/4} - a^{-3/4}}{a_c^{-3/4} - a_P^{-3/4}}\right]^{7/3} \\ &\simeq \frac{8\pi \times 10^{37}}{3} \left(\frac{a}{a_P}\right)^{-1/4}. \end{aligned} \tag{3.9}$$

Equation (3.9) integrates to

$$a(t) = a_P \left[1 + 10^{18} \sqrt{\frac{5\pi}{12}} (t - t_P)\right]^8 \tag{3.10}$$

showing *acceleration* as $\ddot{a} > 0$.

If expansion (3.10) yields sufficient inflation in the early universe,

$$\frac{a_c}{a_P} = 10^{28}. \tag{3.11}$$

The universe comes out of the inflationary phase at $t = t_c$ when $a(t)$ acquires the value a_c . So, from (3.10) and (3.11), it is obtained that

$$t_c \simeq t_P + 10^{-18} \sqrt{\frac{12}{5\pi}} \left[\left(\frac{a_c}{a_P}\right)^{1/8} - 1\right] \simeq 2.76 \times 10^4 t_P \tag{3.12}$$

using (3.11).

For $a \geq a_c$, we have Friedmann equation (3.8) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{B}{a^4}. \tag{3.13}$$

This equation integrates to

$$a(t) = a_c [1 + \sqrt{B}(t - t_c)]^{1/2}. \tag{3.14}$$

Equation (3.14) yields $\ddot{a} < 0$ showing *deceleration* driven by dark radiation term.

4 Deceleration Followed by Acceleration in the Late Universe as Well as Future Collapse of the Universe

In the late universe, the effective Friedmann equation is given by (2.21). Using (2.23)–(2.25) in (2.21), we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{C}{a^3} + \frac{8D}{15}a^{3/4}\left[1 - \frac{5E}{6D}a^{3/4}\right], \tag{4.1a}$$

where

$$D = \frac{a_c^{-3/4}}{4\pi G}\left[1 - \frac{28\pi G}{135}\left(\frac{1}{180\beta}\right)^{1/3}a_c^{-1/4}\right] \tag{4.1b}$$

and

$$E = \frac{a_c^{-3/2}}{4\pi G}\left[1 - \frac{4\pi G}{135}\left(\frac{1}{180\beta}\right)^{1/3}a_c^{-1/4}\right] \tag{4.1c}$$

being obtained from (2.20a) and (2.20b) using (2.23)–(2.25) and (3.2b).

The first term, on r.h.s. of (4.1a), emerges spontaneously and has the form of matter density, so it is recognized as dark matter density like dark radiation. Moreover, the second and third terms on r.h.s. of (4.1a) emerges due to linear and non-linear terms of curvature. It is interesting to see that if

$$\rho_{\text{de}}^{\text{ph}} = \frac{D}{5\pi G}a^{3/4} \tag{4.2}$$

and

$$\lambda = \frac{3D^2}{25\pi GE}, \tag{4.3}$$

(4.1a) looks like

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\frac{3C}{8\pi Ga^3} + \rho_{\text{de}}^{\text{ph}}\left\{1 - \frac{\rho_{\text{de}}^{\text{ph}}}{2\lambda}\right\}\right]. \tag{4.4}$$

Conservation equation (3.5) for $\rho_{\text{de}}^{\text{ph}}$ yields

$$w_{\text{de}}^{\text{ph}} = -\frac{5}{4}. \tag{4.5}$$

Equation (4.5) shows that the curvature-induced energy density $\rho_{\text{de}}^{\text{ph}}$ mimics phantom dark energy as $w_{\text{de}}^{\text{ph}} < -1$. Thus, in the late universe, a phantom model is obtained from curvature without using any source of exotic matter. Apart from this, (4.4) contains a term $-(\rho_{\text{de}}^{\text{ph}})^2/2\lambda$ analogous to brane-gravity correction to the Friedmann equation (FE) for negative brane-tension [15] and modifications in FE due to loop-quantum effects [14]. Here λ is called *cosmic tension* [10–13].

According to WMAP results [23], present density of pressureless dark matter is obtained to be $\rho_0^{(m)} = 0.23\rho_0^{\text{cr}}$ and present dark energy density $\rho_{\text{de}0}^{\text{ph}} = 0.73\rho_0^{\text{cr}}$ with

$$\rho_0^{\text{cr}} = \frac{3H_0^2}{8\pi G},$$

where current Hubble’s rate of expansion $H_0 = 100 \text{ hkm/Mpc second} = 2.32 \times 10^{-42} h \text{ GeV}$ and $h = 0.68$. Thus,

$$\rho_0^{\text{cr}} = 2.9 \times 10^{-47} \text{ GeV}^4. \tag{4.6}$$

Using these values, it is obtained that

$$\rho^{(m)} = \frac{3C}{8\pi G a^3} = \frac{6.67 \times 10^{-48}}{a^3} \tag{4.7}$$

and

$$\rho_{\text{de}}^{\text{ph}} = \frac{D}{5\pi G} a^{3/4} = \frac{2.117 \times 10^{-47}}{a^{3/4}} \tag{4.8}$$

from (4.2).

Connecting (4.4), (4.7) and (4.8), it is obtained that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left[\frac{6.67 \times 10^{-48}}{a^3} + 2.12 \times 10^{-47} a^{3/4} \left\{ 1 - \frac{2.117 \times 10^{-47} a^{3/4}}{2\lambda} \right\} \right]. \tag{4.9}$$

Equation (4.9) shows that

$$\frac{6.67 \times 10^{-48}}{a^3} > 2.117 \times 10^{-47} a^{3/4}$$

for $a < 0.735$ and

$$\frac{6.67 \times 10^{-48}}{a^3} < 2.117 \times 10^{-47} a^{3/4}$$

for $a > 0.735$.

It means that a *transition* from matter-dominance to DE-dominance takes place at

$$a_* = 0.735 \tag{4.10}$$

giving *red-shift*

$$z_* = \frac{1}{a_*} - 1 = 0.3607 \tag{4.11}$$

which is very closed to lower limit of z_* given by 16 Type supernova observations [3]. Thus, for $a < 0.735$, (4.9) is approximated as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left[\frac{6.67 \times 10^{-48}}{a^3} \right] = \frac{5.59 \times 10^{-85}}{a^3}, \tag{4.12}$$

which integrates to

$$a(t) = a_d [1 + 7.48 \times 10^{-43} a_d^{-3/2} (t - t_d)]^{2/3}. \tag{4.13}$$

It shows decelerated expansion as $\ddot{a} < 0$.

When $a \geq 0.735$, (4.9) is approximated as

$$\left(\frac{\dot{a}}{a}\right)^2 = 1.77 \times 10^{-84} a^{3/4} \left[1 - \frac{2.117 \times 10^{-47} a^{3/4}}{2\lambda} \right]. \tag{4.14}$$

Equation (4.14) integrates to

$$a(t) = \left[\frac{2.117 \times 10^{-47}}{2\lambda} + \left\{ \sqrt{a_*^{-3/4} - \frac{2.117 \times 10^{-47}}{2\lambda}} - 5 \times 10^{-43}(t - t_*) \right\}^2 \right]^{-4/3}. \tag{4.15}$$

This scale factor yields $\ddot{a} > 0$ showing *acceleration* in the late universe. In (4.15), t_* is the time of transition from deceleration to acceleration in the late universe.

Using the present age of the universe $t_0 = 13.7 \text{ Gyr} = 6.6 \times 10^{41} \text{ GeV}^{-1}$ and $a_0 = 1$ (as given above) in (4.15), t_* is calculated as

$$t_* = t_0 - 0.4 \times 10^{41} \text{ GeV}^{-1} = 6.2 \times 10^{41} \text{ GeV}^{-1} = 12.86 \text{ Gyr}. \tag{4.16}$$

Equation (4.14) shows that accelerated expansion (4.15) stops at $a = a_e$ satisfying the condition

$$2.117 \times 10^{-47} a_e^{3/4} = 2\lambda. \tag{4.17}$$

$a(t)$, given by (4.15), acquires the value a_e by the time

$$t_e = t_* - 2 \times 10^{42} \left[\sqrt{a_e^{-3/4} - \frac{2.117 \times 10^{-47}}{2\lambda}} - \sqrt{a_*^{-3/4} - \frac{2.117 \times 10^{-47}}{2\lambda}} \right]. \tag{4.18}$$

The Friedmann equation (4.9) shows that, at $t \geq t_e$, expansion of the universe is driven by matter again and it reduces to (4.12) yielding the solution

$$a(t) = a_e [1 + 1.21 \times 10^{-42} a_e^{-3/2} (t - t_e)]^{2/3} \tag{4.19}$$

which shows decelerated expansion. Thus another transition from *acceleration* to *deceleration* will take place at $t = t_e$.

It is interesting to note that the term

$$2.117 \times 10^{-47} a^{3/4} \left[1 - \frac{2.117 \times 10^{-47} a^{3/4}}{2\lambda} \right]$$

will be negative as $a > a_e$. So, gradually universe will reach a state, where scale factor $a(t)$ acquires its maximum value a_m . At $a = a_m$, $\dot{a} = 0$ in (4.9) and a_m satisfies the condition

$$\frac{6.67 \times 10^{-48}}{a_m^3} = 2.117 \times 10^{-47} a_m^{3/4} \left\{ \frac{2.117 \times 10^{-47} a_m^{3/4}}{2\lambda} - 1 \right\}. \tag{4.20}$$

a_m gives the maximum expansion, so for $t > t_m$ universe will change its direction and retrace back leading to contraction. As a consequence, for $t > t_m$, $a(t)$ will decrease and a^{-3} term, in (4.9), will dominate yielding the effective equation

$$\frac{\dot{a}}{a} = - \frac{8.099 \times 10^{-43}}{a^{3/2}}.$$

Here, $\frac{\dot{a}}{a} < 0$ due to contraction. This equation is integrated to

$$a(t) = a_m [1 - 1.21 \times 10^{-42} a_m^{-3/2} (t - t_m)]^{2/3}. \tag{4.21}$$

Equation (4.21) shows that at time

$$t = t_m + 8.26 \times 10^{41} a_m^{3/2}, \quad (4.22)$$

$a = 0$. It means that universe will collapse at this time.

5 Summary

Results, obtained above, are summarized as follows. Here $f(R)$ -gravitational action is obtained by adding higher-order terms R^2 and $R^{(2+r)}$ of scalar curvature R to the Einstein-Hilbert term. Gravitational field equations are derived from this action. Using $R \sim a^{-n}$ in trace of $f(R)$ -gravity field equation, Friedmann equation is obtained. It is found above that if $r = 3$ and $n = 1/4$, FE (obtained here) contains quintessence like dark energy term as well as radiation like term in the early universe. Here, radiation emerges spontaneously and is termed as dark radiation which is analogous to a similar term in brane-gravity-based FE. Dark energy term is induced by curvature and it vanishes when the scale factor $a(t)$ acquires a finite value a_c given by (3.2b). It is interesting to see that, in the late universe, dark matter term emerges spontaneously and in the very late universe (when it is 12.86 Gyrs old) the universe is dominated by curvature-induced phantom dark energy with $w = -1.25$. Contribution of R to DE is a physical concept in addition to its usual role as a geometrical field. Thus, dual roles of R (as a physical field as well as a geometrical field) [24–34] are manifested here. The cosmological scenario, obtained here, from $f(R)$ -gravity with higher-order terms R^2 and R^5 , is given as follows.

It is found that the early universe inflated for a very short period with power-law speeded-up expansion, driven by curvature-induced quintessence DE. When the scale factor increased upto 10^{28} times the scale factor at Planck scale, quintessence DE density vanished. As a consequence, universe came out of the inflationary era. Later on, early universe decelerated as $t^{1/2}$ driven by dark radiation. Subsequently, when the universe became sufficiently old, it decelerated as $t^{2/3}$ driven by dark matter. At red-shift $z_{**} = 0.3607$, a transition from deceleration to acceleration took place and universe began to accelerate driven by curvature-induced phantom DE explaining the present acceleration of the universe. It is found that the late acceleration is *transient* and it stops when phantom DE grows to a finite value equal to 2λ . Here λ is the cosmic tension analogous to brane-tension, which is explained above. Interestingly, the phantom model (obtained here) is free from the menace of future-singularity. Here, it is shown that universe will reach its maximum expansion at a time t_m . Later on, it will contract and collapse in a finite future time. Thus, it is found that contrary to $f(R)$ -dark energy models, we get a viable cosmology from $f(R)$ -gravity. Also, it is found that curvature induced phantom DE has a crucial role in the dynamics of present and future universe giving different phases mentioned above. Results, obtained above, support observations made so far.

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